EXISTANCE AND UNIQUENESS THEORY. WRONSKIAN.

NOTE:-1

TABLE-1

ROOTS	$y''+ay'+by=0$	$x^2y''+axy'+by=0$
Real and unequal	$y=C_1e^{\lambda_1 x}+C_2e^{\lambda_2 x}$ λ_1, λ_2	$y=C_1x^{m_1}+C_2x^{m_2}$ m_1, m_2
Real and equal	$y=(C_1+C_2x)e^{\lambda x}$ λ, λ	$y=(C_1+C_2\ln x)x^{m}$ m, m
Complex and Conjugate	$y=e^{\alpha x}(C_1\cos\beta x+C_2\sin\beta x)$ $\alpha\pm i\beta$	$y=x^{\alpha}(C_1\cos(\beta\ln x)+C_2\sin(\beta\ln x))$ $\alpha\pm i\beta$

NOTE:-2

The general solution of a second order DE is always in the form of

$$\boxed{y = C_1y_1 + C_2y_2}$$

where, y_1 and y_2 are called the basis of solutions

NOTE:-3

The Wronskian (W) of a basis of solutions y_1 and y_2 is given by

$$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

NOTE:-4

(i) If the Wronskian $W=f(x)$ then the basis of solutions y_1 and y_2 are <u>linearly independent</u>

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(11) If the Wronskian W = a constant then the basis of solutions y_1 and y_2 are linearly dependent

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NO-1-TO-8

Find the Wronskian of the given basis of solutions

2. $1, e^x$

Let $y_1 = 1$ $\quad y_2 = e^x$

$y_1' = 0$ $\quad y_2' = e^x$

The Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 1 & e^x \\ 0 & e^x \end{vmatrix} = e^x - 0 = e^x$

5. $x^4, x^4 \ln x$

Let $y_1 = x^4$ $\quad y_2 = x^4 \ln x$

$y_1' = 4x^3$ $\quad y_2' = 4x^3 \ln x + x^3$

The Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^4 & x^4 \ln x \\ 4x^3 & 4x^3 \ln x + x^3 \end{vmatrix}$

$= 4x^7 \ln x + x^7 - 4x^7 \ln x$

$= x^7.$

9-TO-17

Find a second order homogeneous linear DE for which the given functions are solutions. Find the Wronskian and use it to verify linear independence

9. e^{3x}, xe^{3x}

Let $y_1 = e^{3x}$ $\quad y_2 = xe^{3x}$

since these are the basis of solutions, The general solution is $y = c_1 y_1 + c_2 y_2$

$\Rightarrow y = c_1 e^{3x} + c_2 x e^{3x} = (c_1 + c_2 x) e^{3x}$

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So, the roots of the auxilary equation are 3, 3

The factors of the auxilary equation are $\lambda-3, \lambda-3$

The auxilary equation is

$$(\lambda-3)(\lambda-3)=0$$

$$\Rightarrow \lambda^2-6\lambda+9=0$$

Thus the DE is $y''-6y'+9y=0$.

Now, $y_1' = 3e^{3x}$ $\quad y_2' = e^{3x}+3xe^{3x}$

The wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$$= \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x}+3xe^{3x} \end{vmatrix}$$

$$= e^{6x}+3xe^{6x}-3xe^{6x}$$

$$= e^{6x} = f(x)$$

So, the basis of solutions are linearly independent.

14. $1, e^{-2x}$

Let $y_1 = 1$ $\quad y_2 = e^{-2x}$

Since these are the basis of solutions. The general solution is $y = c_1y_1 + c_2y_2$

$$\Rightarrow y = c_1 1 + c_2 e^{-2x}$$

$$\Rightarrow y = c_1 e^{0x} + c_2 e^{-2x}$$

So, the roots of the auxilary equation are 0, -2

The factors of the auxilary equation are $\lambda-0, \lambda+2$

The auxilary equation is

$$(\lambda-0)(\lambda+2)=0$$

$$\Rightarrow \lambda^2+2\lambda=0$$

Thus the DE is $y''+2y'=0$

now $y_1' = 0$ $\quad y_2' = -2e^{-2x}$

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The wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$$= \begin{vmatrix} 1 & e^{-2x} \\ 0 & -2e^{-2x} \end{vmatrix} = -2e^{-2x} = f(x)$$

So, the basis of solutions are linearly independent.

15. $\cos 2\pi x, \sin 2\pi x$

Let $y_1 = \cos 2\pi x \qquad y_2 = \sin 2\pi x$

Since these are the basis of solutions. The general solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \cos 2\pi x + c_2 \sin 2\pi x$$

$$= e^{0x}(c_1 \cos 2\pi x + c_2 \sin 2\pi x)$$

So, the roots of the auxilary equation are

$$0 \pm 2\pi i$$

The factors of the auxilary equation are

$$\lambda - 2\pi i, \quad \lambda + 2\pi i$$

The auxilary equation is

$$(\lambda - 2\pi i)(\lambda + 2\pi i) = 0$$

$$\Rightarrow \lambda^2 - 4\pi^2 i^2 = 0$$

$$\Rightarrow \lambda^2 + 4\pi^2 = 0$$

Thus the DE is $y'' + 4\pi^2 y = 0$

Now, $y_1' = -2\pi \sin 2\pi x \qquad y_2' = 2\pi \cos 2\pi x$

The wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$$= \begin{vmatrix} \cos 2\pi x & \sin 2\pi x \\ -2\pi \sin 2\pi x & 2\pi \cos 2\pi x \end{vmatrix}$$

$$= 2\pi \cos^2 2\pi x + 2\pi \sin^2 2\pi x$$

$$= 2\pi = \text{const.}$$

So, the basis of solutions are linearly dependent.

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10. x^5, x^{-5}

Let $y_1 = x^5 \qquad y_2 = x^{-5}$

since these are the basis of solutions, the general solution is

$$y = c_1 y_1 + c_2 y_2 = c_1 x^5 + c_2 x^{-5}$$

So, the roots of the auxilary equation are $5, -5$

The factors of the auxilary equation are

$$m-5, \; m+5$$

The auxilary equation is

$$(m-5)(m+5) = 0$$

$$\Rightarrow m^2 - 25 = 0$$

$$\Rightarrow m(m-1) + m - 25 = 0$$

Thus, the DE is $x^2 y'' + x y' - 25 y = 0$

Now $y_1' = 5x^4 \qquad y_2' = -5x^{-6}$

The wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$$= \begin{vmatrix} x^5 & x^{-5} \\ 5x^4 & -5x^{-6} \end{vmatrix} = -5x^{-1} - x^{-1}$$

$$= -6x^{-1}$$

$$= f(x)$$

So, the basis of solutions are linearly independent.

11. $x^2, \; x^2 \ln x$

Let $y_1 = x^2 \qquad y_2 = x^2 \ln x$

since these are the basis of solutions the general solution is

$$y = c_1 y_1 + c_2 y_2 = c_1 x^2 + c_2 x^2 \ln x = (c_1 + c_2 \ln x) x^2$$

So, the roots of the auxilary equation are $2, 2$

The factors of the auxilary equation are

$$m-2, \; m-2$$

The auxilary equation is

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$$(m-2)(m-2)=0$$

$$\Rightarrow m^2-4m+4=0$$

$$\Rightarrow m(m-1)-3m+4=0$$

The DE is $x^2y''-3xy'+4y=0$

Now $y_1' = 2x$ $\quad y_2' = 2x\ln x + x$

The Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^2\ln x \\ 2x & 2x\ln x + x \end{vmatrix}$

$$= 2x^3\ln x + x^3 - 2x^3\ln x$$

$$= x^3 = f(x)$$

So, the basis of solutions are linearly independent.

16. $\cos(\ln x)$, $\sin(\ln x)$

Let $y_1 = \cos(\ln x)$ $\quad y_2 = \sin(\ln x)$

Since these are the basis of solutions the general solution is

$$y = c_1y_1 + c_2y_2 = c_1\cos(\ln x) + c_2\sin(\ln x)$$

$$= x^0\,(c_1\cos(\ln x) + c_2\sin(\ln x))$$

So, the roots of the auxiliary equation are

$$0+i,\ 0-i$$

The factors of the auxiliary equation are

$$m-i,\ m+i$$

The auxiliary equation is

$$(m-i)(m+i) = 0$$

$$\Rightarrow m^2 - i^2 = 0$$

$$\Rightarrow m^2 + 1 = 0$$

$$\Rightarrow m(m-1) + m + 1 = 0$$

The differential equation is

$$x^2y'' + xy' + y = 0$$

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$$y_1' = -\frac{1}{x}\sin(\ln x) \qquad y_2' = \frac{1}{x}\cos(\ln x)$$

The Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$$= \begin{vmatrix} \cos(\ln x) & \sin(\ln x) \\ -\frac{1}{x}\sin(\ln x) & \frac{1}{x}\cos(\ln x) \end{vmatrix}$$

$$= \frac{1}{x}\cos^2(\ln x) + \frac{1}{x}\sin^2(\ln x)$$

$$= \frac{1}{x} = f(x)$$

So, the basis of solutions are linearly independent

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